MATHCOUNTS® Minis September 2014 Activity Solutions

Warm-Up!

1. We are asked to determine the value of the sum 1 + 2 + 3 + ⋯ + 98 + 99. Adding pairs of these addends, we notice a pattern. For example, pairing the first and last numbers, we have 1 + 99 = 100. Then pairing the second number with the next to last number, we see that 2 + 98 = 100. We will be able to do this for a total of 49 pairs of addends, with the addend of 50 left in the middle unpaired. That means the sum of the first 99 positive integers is 49(100) + 50 = 4900 + 50 = **4950**.

2. Figure 1 has 1 dot. Figure 2 has 3 dots, which is 2 more that the previous figure. Figure 3 has 6 dots, which is 3 more than the previous figure. Finally, Figure 4 has 10 dots, which is 4 more than the previous figure. Notice the pattern shown in the table below. Therefore, Figure 10 has 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = **55** dots. The numbers of dots in each figure form a sequence of numbers commonly referred to as the Triangular Numbers. There is a formula to determine the sum of the first *n* positive integers. It is 1 + 2 + 3 + 4 + ⋯+ *n* = *n*(*n* + 1)/2. So, in this case, the number of dots in Figure 10, which represents the tenth Triangular Number, is 10(11)/2 = 110/2 = **55** dots.

FIGURE	DOTS
1	1
2	1 + 2 = 3
3	1 + 2 + 3 = 6
4	1 + 2 + 3 + 4 = 10
⋮	⋮
n	1 + 2 + 3 + 4 + ⋯+ *n*

3a. Because these are each arithmetic sequences, we know the difference between consecutive terms for each sequence remains constant. (In other words, the same amount is added to each term to get the next term.) For the first sequence, we have 11 – 5 = 6, so the common difference is 6. The terms are 5, 11, *11 + 6 or* **17**, *17 + 6 or* **23**, *23 + 6 or* **29**.

3b. Again, the common difference is 6, but we must work backwards. __, __, *5 – 6 or* **–1**, 5, 11. Then __, *–1 – 6 or* **–7**, **–1**, 5, 11. And finally *–7 – 6 or* **–13**, **–7**, **–1**, 5, 11.

3c. Going from 5 to 11, we must add the common difference to 5 a total of four times. The total difference is 6. Dividing this into four equal parts, we see the common difference of the arithmetic sequence is 6/4 = 3/2 = 1.5. Therefore, the sequence is 5, **6.5**, **8**, **9.5**, 11.

4a. One million is 1,000,000. Granted, we're not supposed to be writing anything, but just looking at this, I can see that 1,000,000 = 1000^2, so the square of **999** (or 999^2) would be the largest square less than one million.

4b. This means we need the smallest positive three-digit integer that is 1 more than a multiple of 7. Let's build our smallest positive three-digit multiple of 7... it must be of the form 10_. Dividing 10 by 7 leaves a remainder of 3, and then making the units digit a 5 would make the situation such that 7 now evenly divides into 35. So 105 is the smallest positive three-digit multiple of 7 and **106** is the answer to the original question. (You may have gone a different route... many of us know 77 is a multiple of 7; and if we continue to add 7 or multiples of 7 we can find the number we're looking for. Adding 21 to 77 gives us 98, so 99 would give us a remainder of 1, but isn't a three-digit number; and then adding 7 more we get 106, so this is the smallest positive three-digit integer that is one more than a multiple of 7.)

4c. This is similar to the previous question. I know 999 is a multiple of 9, and that's pretty close to a four-digit number. In fact, if we add 5, we get **1004**, and we know that 1004 will leave a remainder of 5 when it's divided by 9, so this is our answer.

The Problem is solved in the MATHCOUNTS® Mini video.

Follow-up Problems

5. Since $S(19)$ is the sum of the first 19 positive integers, it follows that $S(20) = S(19) + 20$. Therefore, $S(20) - S(19) =$ **20**.

6. When the dots are connected, triangles are created, as shown. Instead of looking at the line segments, let's look at the number of shaded triangles in each figure.

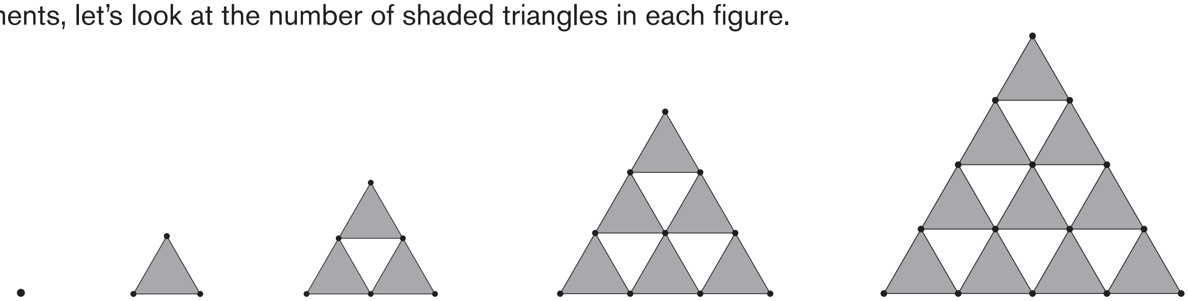

Figure 1 Figure 2 Figure 3 Figure 4 Figure 5

Figure 1 has no triangles. Figures 2, 3, 4 and 5 each have 1, 3, 6 and 10 shaded triangles, respectively. The sequence representing the number of shaded triangles in each figure is 0, 1, 3, 6, 10, ... Recall, that the sequence representing the number of dots in each figure is 1, 3, 6, 10, 15, ... Notice that these two sequences are the same, except the terms are one-off because the first term of the sequence representing the shaded triangles is 0. It follows, then, that in Figure 10 there are $0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$ shaded triangles. Each shaded triangle has a perimeter of 3. That means the length of all the segments in Figure 10 is $45 \times 3 =$ **135** units.

7. The first term in Jenny's list is $1^2 = 1$. The product of the first two terms in her list must be $2^2 = 4$. Therefore, the second term is $4 \div 1 = 4$. The product of the first three terms must be $3^2 = 9$. So, the third term is $9 \div (1 \times 4) = 9/4$. The product of the first four terms must be $4^2 = 16$. It follows, then, that the fourth term is $16 \div (1 \times 4 \times (9/4)) = 16/9$. Notice the pattern? The nth term in Jenny's list equals $n^2/(n - 1)^2$. So, the last term in Jenny's list, which is the 12th term, equals $12^2/11^2 =$ **144/121**.

8. We are given a formula for the sum of the first n terms of a sequence and then asked to determine the value of a_6. The sum of the first six terms is $S_6 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = S_5 + a_6$, so $S_6 - S_5 = a_6$. Using the formula we can calculate S_5 and S_6 we get $S_5 = 5^2 + 4 \times 5 + 8 = 25 + 20 + 8 = 53$ and $S_6 = 6^2 + 4 \times 6 + 8 = 36 + 24 + 8 = 68$. Substituting these values, we see that $a_6 = S_6 - S_5 = 68 - 53 =$ **15**.